Jackson 3.9

The boundary condition $\Phi|_{z=0}=0$, $\Phi|_{z=L}=0$ demands modification of separation of variables using modified Bessel functions, as prescribed by Jackson (pg 116).

Specifically, in $\Phi(\rho,\phi,z) = R(\rho)Q(\phi)Z(z)$, we modify the ansatz via

$$\begin{cases} Z'' - k^2 Z = 0 \\ Q'' + \nu^2 Q = 0 \\ R'' + \frac{1}{\rho}R' + (k^2 - \frac{\nu^2}{\rho^2})R = 0 \end{cases} \Rightarrow \begin{cases} Z'' + k^2 Z = 0 \\ Q'' + \nu^2 Q = 0 \\ R'' + \frac{1}{\rho}R' - (k^2 + \frac{\nu^2}{\rho^2})R = 0 \end{cases}$$

The general solutions of Q, Z are $Z \propto e^{\pm ikz}$, $Q \propto e^{\pm i\nu\phi}$, boundary condition and single-valuedness would further demand ν to be integer, k be an integer multiple of $\frac{\pi}{L}$, so we let these integers be indexed by n, m, and we have

$$Q_m(\phi) = A\sin m\phi + B\cos m\phi$$
$$Z_n(z) = E\sin(\frac{n\pi}{L}z)$$

These will give differential equation for R:

$$R'' + \frac{1}{\rho}R' - \left(\left(\frac{n\pi}{L}\right)^2 + \frac{m^2}{\rho^2}\right)R = 0.$$

Jackson (pg 116) gives the solution for R of this form via the modified Bessel functions:

$$R(\rho) = C\, I_m\left(\frac{n\pi}{L}\rho\right) + D\, K_m\left(\frac{n\pi}{L}\rho\right)$$

where $I_m(x) = i^{-m} J_m(ix)$, $K_m(x) = \frac{\pi}{2} i^{m+1} H_m^{(1)}(ix)$

This gives the series representation:

$$* \quad \Phi(\rho,\phi,z) = \sum_{n,m} \left[A_{nm} \sin m\phi + B_{nm} \cos m\phi\right] \sin\left(\frac{n\pi}{L} z\right) \left[C_{nm} I_m\left(\frac{n\pi}{L}\rho\right) + D_{nm} K_m\left(\frac{n\pi}{L}\rho\right)\right]$$

For m integer, $H_m^{(1)}$ and J_m are linearly dependent, thus I_m and K_m are linearly dependent. Absorbing coefficients, and removing linear terms, we have the series representation

$$* \quad \Phi(\rho,\phi,z) = \sum_{n,m} \left[A_{nm} \sin m\phi + B_{nm} \cos m\phi\right] \sin\left(\frac{n\pi}{L} z\right) I_m\left(\frac{n\pi}{L}\rho\right)$$

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